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Advanced graph theory. (English) Zbl 1568.05003

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The author describes the book in the preface as being "based on the curriculum of undergraduate and postgraduate courses of universities in India and abroad". This is borne out well in the selection of topics in this textbook. Each of the ten chapters consists of brief discussions on some facet of graph theory or combinatorics and concludes with some worked examples, a bulleted summary of the chapter, a short set of exercises, and suggested readings. Each chapter also opens with a short biographical sketch of a mathematician whose work is relevant to the discussion in that chapter.

Chapter 1 introduces basic graph theoretic terminology, as well as various commonly studied graph classes and graph operations.

Chapter 2 is on trees. The author presents a proof of Cayley's formula for the number of trees on n vertices by showing that this set is in bijective correspondence with the set of functions from $[n]$ to $[n]$ that fix 1 and n. He also discusses the BFS and DFS algorithms to find spanning trees, followed by Kruskal's and Prim's algorithms for finding minimal spanning trees in weighted graphs, and Dijkstra's and Floyd-Warshall's algorithms for finding shortest paths in weighted graphs.

Chapter 3 is on planar graphs. Kuratowski's theorem, Euler's formula, duals of planar graphs, and outerplanar graphs are discussed.

Chapter 4 is on directed graphs. The author proves the theorems of Gallai-Hasse-Roy-Vitaver on the chromatic number of digraphs, of Moon, Dirac, and Ghouila-Houri on directed cycles in tournaments, of Stanley on the number of acyclic orientations of a graph, and of Robbins on di-orientable graphs. Lastly, the max-flow min-cut theorem is proved, and conjectures and results on nowhere zero flows are discussed.

Chapter 5 is on matchings and coverings. Berge's theorem on M-augmenting paths, König's theorem on maximum matchings and minimum vertex covers, and Hall's marriage theorem are proved. The author also discusses perfect matchings, factor-critical graphs, complete matching, and path covers, including the Gallai-Milgram theorem.

Chapter 6 is on graph colorings; the chromatic polynomial is introduced, and so are list colorings. König's theorem and Vizing's theorem on the edge-chromatic number are also proved. The four color problem for planar graphs is introduced, and the five color theorem is proved using Kempe chains. The applications cover scheduling problems.

Chapter 7 is on Ramsey's theory. The author gives a constructive proof of the upper bound $R(r) \leq 2^{2r-3}$ and a probabilistic proof of the lower bound $R(r) > 2^{r/2}$, where $R(r)$ is the least integer k such that every 2-coloring of a graph on k vertices contains either an independent set or a clique of size r. The Ramsey problem is also reformulated in terms of finding monochromatic subsets of colorings of some set. The finite Ramsey theorem is proved from the infinite version using König's lemma. The author also proves the induced Ramsey theorem due to Deuber, Erdős-Hajnal-Pósa and Rödl. The chapter concludes with proofs of Schur's theorem, the Ramsey multiplicity theorem, and the classical bound $R(p,q) \leq \binom{p+q-2}{p-1}$.

Chapter 8 is on enumeration and Pólya counting. The first half of the chapter explains labelled and unlabelled counting problems and generating functions are introduced with examples based on the Catalan, Stirling, and Bell numbers. The second half of the chapter proves Pólya's enumeration theorem and Burnside's lemma.

Chapter 9 discusses some spectral properties of regular graphs, and includes a proof of Sachs's theorem.

Chapter 10 is titled "Emerging trends in graph theory" and discusses the perfect graph theorem, the characterizations of chordal graphs by Gavril and by Rose-Lueker-Tarjan, perfectly orderable graphs, imperfect graphs, the strong perfect graph conjecture, hereditary families of graphs, matroids (including proofs of the matroid intersection and matroid union theorems), and Ramanujan graphs.

The book concludes with a list of references and a brief index.

The textbook is written keeping in mind a typical student at an Indian university. The topics covered in courses on graph theory and combinatorics at most Indian universities will be found in this textbook and more. The title "Advanced graph theory" is a bit of a misnomer, since the book starts right from the basics, and also covers topics in the theory of enumeration. There is an abundance of pictures and solved examples to assist the reader in every chapter.

The book has the potential to be a useful resource for a student looking for a one-stop introduction to topics in various subareas of graph theory. Unfortunately, it suffers from drawbacks that prevent this reviewer from offering an unreserved recommendation. The topics within each chapter have clearly been designed to be modular but with limited success. Consequently, the expositions start or end abruptly in many places. There are also innumerable typos, found on nearly every page, many of them having to do with poor transcriptions of mathematical symbols. This makes reading the book quite exhausting.

The exposition in Chapter 1 is particularly inconsistent in what it assumes of the reader. For instance, equivalence relations on sets are gently introduced over several pages, yet the discussion on the random graph model $G(n,p)$ is closed in half a page by assuming all the required background from probability theory. The discussion on equivalence relations is also duplicated: it appears in Section 1.2.5 and again later in Section 1.3. A more severe issue is that the "Handshaking lemma" is first stated and proved only in the middle of Section 1.3 despite being used without comment in the proof of Theorem 1.1 in Section 1.2.4 earlier. Example 1.11 in Section 1.16 purports to show that two graphs on five vertices are isomorphic, despite that they have different numbers of edges. Exercise 1 does not have enough information for it to be solved. Vertex and edge cuts are not defined anywhere but appear in several places starting from Section 3.5. The definition of k-outerplanar graphs in Section 3.8.1 is incorrect; presumably a recursive definition was intended. Several unexplained occurrences of the symbol "X" should usually be reinterpreted as the symbol "χ" for the chromatic number. The notation "$\pi_k(G)$" does not seem to have been introduced before the statement of Theorem 6.5, and does not seem to occur anywhere after its proof. The title of Chapter 7 in the Table of Contents is incorrect. All the exercises are uniformly formatted, though the author mentions in the preface that the difficult exercises are "bold-faced and highlighted".

In summary, the textbook covers a decent selection of topics in less than 300 pages, and effort has clearly been put into making it accessible to undergraduate students. However, it urgently needs another thorough round (or two) of proofreading.

Reviewer: Brahadeesh Sankarnarayanan (Mumbai)

MSC:

05-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to combinatorics
05C05 Trees
05C10 Planar graphs; geometric and topological aspects of graph theory
05C20 Directed graphs (digraphs), tournaments
05C21 Flows in graphs
05C70 Edge subsets with special properties (factorization, matching, partitioning, covering and packing, etc.)
05C15 Coloring of graphs and hypergraphs
05C55 Generalized Ramsey theory
05A15 Exact enumeration problems, generating functions
05C50 Graphs and linear algebra (matrices, eigenvalues, etc.)
05C17 Perfect graphs
05B35 Combinatorial aspects of matroids and geometric lattices

Keywords:

trees; planar graphs; tournaments; matchings; graph colorings; Ramsey theory; Pólya's enumeration theorem; spectral graph theory; perfect graphs; chordal graphs; matroids

Full Text: DOI